Generating Real-Valued OFDM Signals with the Discrete Fourier Transform

Steve C. Thompson*

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Abstract

For some applications, a real-valued OFDM signal is required. This can be done by taking a DFT of a conjugate symmetric vector. The spectral efficiency of the real-valued OFDM signal is the same as the spectral efficiency of the complex-valued OFDM signal.

1 Signal Description

The baseband OFDM signal is typically written as

$$x(t) = \sum_{k=0}^{N-1} X_k e^{j2\pi kt/T}, \quad 0 \le t < T, \tag{1}$$

where N is the number of subcarriers, $\{X_k\}_{k=1}^{N-1}$ are the data symbols and T is the block period. Sampling $x(t)$ at N equally spaced intervals over $0 \le t < T$ yields the sequence,

$$x[n] = x(t)|_{t=nT/N} = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1, \tag{2}$$

which is the inverse discrete Fourier transform (IDFT) of the vector $\mathbf{X} = [X_0, X_1, \ldots, X_{N-1}]$. The sequence is complex-valued in general. However it can be made real-valued by making $\mathbf{X}$ conjugate symmetric:

$$X_{N/2+k} = X^*_{N/2-k}, \quad k = 1, 2, \ldots, N/2-1, \tag{3}$$

and

$$X_0 = X_{N/2} = 0. \tag{4}$$

*With the Center for Wireless Communications, Department of Electrical and Computer Engineering, University of California at San Diego, La Jolla, CA. Feedback is welcome, contact Steve at sct@ucsd.edu. If you find this useful please reference as: S. C. Thompson, "Generating Real-Valued OFDM Signals with the Discrete Fourier Transform." [Online]. Available: http://zeidler.ucsd.edu/∼sct/pubs/t5.pdf.

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The IDFT is then

$$
\begin{aligned}
x[n] &= \sum_{k=1}^{N-1} X_k e^{j2\pi kn/N} \\
&= \sum_{k=1}^{N/2-1} X_{N/2-k} e^{j2\pi(N/2-k)n/N} + X_{N/2+k} e^{j2\pi(N/2+k)n/N} \\
&= \sum_{k=1}^{N/2-1} X_{N/2-k} e^{j2\pi(N/2-k)n/N} + X^*_{N/2-k} e^{j2\pi(N/2+k)n/N},
\end{aligned}
\tag{5}
$$

$n = 0, 1, \ldots, N-1$. But since

$$
\begin{aligned}
e^{j2\pi(N/2+k)n/N} &= e^{j2\pi(N/2+k)n/N} e^{-j2\pi Nn/N} \\
&= e^{j2\pi(-N/2+k)n/N} \\
&= e^{-j2\pi(N/2-k)n/N},
\end{aligned}
\tag{6}
$$

(5) can be written as

$$
x[n] = \sum_{k=1}^{N/2-1} X_{N/2-k} e^{j2\pi(N/2-k)n/N} + X^*_{N/2-k} e^{-j2\pi(N/2-k)n/N}, \tag{7}
$$

$n = 0, 1, \ldots, N-1$. Using the identity $A + A^* = 2\Re\{A\}$,

$$
\begin{aligned}
x[n] &= 2\Re\left\{ \sum_{k=1}^{N/2-1} X_{N/2-k} e^{j2\pi(N/2-k)n/N} \right\} \\
&= 2\Re\left\{ \sum_{k=1}^{N/2-1} X_k e^{j2\pi kn/N} \right\}, \quad n = 0, 1, \ldots, N-1.
\end{aligned}
\tag{8}
$$

And since $\Re\{AB\} = \Re\{A\}\Re\{B\} - \Im\{A\}\Im\{B\}$,

$$
x[n] = 2 \sum_{k=1}^{N/2-1} \Re\{X_k\} \cos(2\pi kn/N) - \Im\{X_k\} \sin(2\pi kn/N), \tag{9}
$$

$n = 0, 1, \ldots, N-1$. Thus, $x[n]$ is real. Passing the sequence through a D/A converter yields the continuous-time real-valued OFDM signal:

$$
x(t) = 2 \sum_{k=1}^{N/2-1} \Re\{X_k\} \cos(2\pi kt/T) - \Im\{X_k\} \sin(2\pi kt/T). \tag{10}
$$

Now, suppose the data symbols are derived from a M-QAM (quadrature-amplitude modulation) constellation; that is,

$$
X_k = \Re\{X_k\} + j\Im\{X_k\}, \tag{11}
$$

where

$$
\Re\{X_k\}, \Im\{X_k\} \in \{\pm 1, \pm 3, \ldots, \pm(\sqrt{M} - 1)\}, \quad \text{for all } k. \tag{12}
$$

In other words, the real and imaginary components are derived from $\sqrt{M}$-PAM (pulse-amplitude modulation) constellations. Therefore, processing M-QAM data with the IDFT, (10) is a real-valued $\sqrt{M}$-PAM OFDM signal.

2 Spectral Efficiency

Complex-valued baseband signals are transmitted as bandpass signals, centered at a carrier frequency f_c Hz. This is the case for the complex-valued signal in (1). The transmitted signal is represented as

$$s_1(t) = \Re\left\{x(t)e^{j2\pi f_\mathrm{c} t}\right\}. \tag{13}$$

In the frequency domain, $x(t)$ is shifted to the right by f_c Hz, and the subcarriers are centered at $f_\mathrm{c}, f_\mathrm{c} + 1/T, f_\mathrm{c} + 2/T, \ldots, f_\mathrm{c} + (N-1)/T$ Hz. The effective bandwidth of the signal is therefore N/T Hz. Each data symbol represents $\log_2 M$ bits (i.e., they are assumed to be selected from a M-ary constellation), therefore the spectral efficiency is

$$\mathcal{S}_1 = \frac{\text{Bits per second (b/s)}}{\text{Bandwidth (Hz)}} = \frac{N \log_2 M/T}{N/T} = \log_2 M \ \ \text{b/s/Hz}. \tag{14}$$

The real-valued OFDM signal in (10) has the same spectral efficiency as the complex-valued signal, so long as it is transmitted at baseband. Transmitting the signal as-is, $\Re\{X_k\}$, $k = 1, 2, \ldots, (N/2) - 1$, modulate cosine subcarriers centered at $1/T, 2/T, \ldots, (N/2-1)/T$ Hz; and likewise, $\Im\{X_k\}$, $k = 1, 2, \ldots, N/2 - 1$, modulate sine subcarriers at the same frequencies. The effective bandwidth of the signal is $(N/2-1)/T$ Hz[1], and since the real and imaginary parts of M-QAM X_k represent $0.5 \log_2 M$ bits, the spectral efficiency of the real-valued OFDM signal is

$$\mathcal{S}_2 = \frac{\text{Bits per second (b/s)}}{\text{Bandwidth (Hz)}} = \frac{2 \times (N/2-1) 0.5 \log_2 M/T}{(N/2-1)/T} = \log_2 M \ \ \text{b/s/Hz}. \tag{15}$$

Therefore the spectral efficiency is the same as for the complex case.

However, the spectral efficiency of the real-valued signal is 1/2 that of the complex-valued signal if the real-valued signal is translated up to a carrier frequency. This is due to the fact that the cosine and sine subcarriers in (10) have a double sideband spectrum: i.e., $\cos(2\pi kt/T)$ [or $\sin(2\pi kt/T)$] has a spectral components at $\pm k/T$ Hz. [This isn't the case for the complex-valued signal, which has complex sinusoids: $\exp(j2\pi kt/T)$ has a spectral component only at k/T Hz and is thus considered single sideband.] The carrier frequency is typically much larger than the signal bandwidth, so the frequency translation brings all the negative frequencies to the positive side: $-(N/2-1)/T + f_\mathrm{c} \gg 0$. Consequently, the passband transmission of (10) results in a signal with double the bandwidth and 1/2 the spectral efficiency.

[1] Only the positive frequencies, $f \geq 0$, count.